Mathematics Proof Portfolio

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A curated collection of classical results with complete proofs.
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1. Number Theory

Definition 1 (Rational and Irrational Numbers)

A real number r is called *rational* if it can be written as $r = p/q$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. Otherwise r is called *irrational*.

Theorem 2 (Irrationality of $\sqrt{2}$)

The number $\sqrt{2}$ is irrational.

Proof. Suppose for contradiction that $\sqrt{2}$ is rational. Then we can write

$$\sqrt{2} = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$ (the fraction is in lowest terms). Squaring both sides gives

$$2q^2 = p^2.$$

Hence p^2 is even, which forces p to be even (since if p were odd, p^2 would be odd). Write $p = 2k$ for some $k \in \mathbb{Z}$. Then

$$2q^2 = (2k)^2 = 4k^2 \implies q^2 = 2k^2,$$

so q^2 is even, and again q must be even. But then both p and q are divisible by 2, contradicting $\gcd(p, q) = 1$. Therefore $\sqrt{2}$ is irrational. ∎

Theorem 3 (Infinitude of Primes (Euclid, ~300 BC)**)**

There are infinitely many prime numbers.

Proof. Suppose for contradiction that there are only finitely many primes, say $p_1, p_2, \ldots, p_n$. Define

$$N = p_1 \cdot p_2 \cdots p_n + 1.$$

Since $N > 1$, it has at least one prime divisor p. But for each $i \in \{1, \ldots, n\}$, dividing N by p_i leaves remainder 1, so $p_i \nmid N$. Thus $p \notin \{p_1, \ldots, p_n\}$, contradicting the assumption that the list was complete. Hence no finite list can contain all primes. ∎

2. Combinatorics and Induction

Theorem 1 (Sum of First n Positive Integers)

For every positive integer n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Proof. We proceed by *mathematical induction* on n.

Base case $(n = 1)$: $\sum_{k=1}^{1} k = 1 = \frac{1 \cdot 2}{2}$. ✓

Inductive step: Assume the formula holds for some $n = m \geq 1$, i.e.,

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}.$$

We must show it holds for $n = m + 1$. Indeed,

$$\begin{aligned}
\sum_{k=1}^{m+1} k &= \left(\sum_{k=1}^{m} k\right) + (m+1) \\
&= \frac{m(m+1)}{2} + (m+1) \\
&= (m+1)\left(\frac{m}{2} + 1\right) \\
&= \frac{(m+1)(m+2)}{2},
\end{aligned}$$

which is the formula for $n = m + 1$. By the principle of mathematical induction the result holds for all $n \in \mathbb{Z}^+$. ■

Corollary 2 (Sum of First n Odd Integers)

For every positive integer n,

$$\sum_{k=1}^{n} (2k-1) = n^2.$$

Proof. Using the theorem above,

$$\sum_{k=1}^{n} (2k-1) = 2\sum_{k=1}^{n} k - n = 2 \cdot \frac{n(n+1)}{2} - n = n(n+1) - n = n^2.$$

■

3. Real Analysis

Definition 1 (Limit of a Sequence)

A sequence $(a_n)_{n\geq 1}$ of real numbers *converges* to $L \in \mathbb{R}$, written $\lim_{n\to\infty} a_n = L$, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > N \implies |a_n - L| < \varepsilon.$$

Proposition 2 (Convergence of a Geometric Sequence)

Let $r \in \mathbb{R}$ with $|r| < 1$. Then $\lim\limits_{n\to\infty} r^n = 0$.

Proof. If $r = 0$ the result is immediate. Assume $0 < |r| < 1$ and let $\varepsilon > 0$ be given. We seek N such that $n > N$ implies $|r|^n < \varepsilon$.
Since $|r| < 1$, we have $1/|r| > 1$, so we may write $1/|r| = 1 + h$ for some $h > 0$. By Bernoulli's inequality,

$$\left(\frac{1}{|r|}\right)^n = (1+h)^n \geq 1 + nh \ > \ nh,$$

hence

$$|r|^n = \frac{1}{(1+h)^n} < \frac{1}{nh}.$$

Choose $N = \lceil 1/(h\varepsilon) \rceil$. Then for all $n > N$,

$$|r^n - 0| = |r|^n < \frac{1}{nh} < \frac{1}{Nh} \leq \varepsilon.$$

Therefore $r^n \to 0$ as $n \to \infty$. ∎

Lemma 3 (Squeeze Lemma)

Let (a_n), (b_n), (c_n) be sequences with $a_n \leq b_n \leq c_n$ for all sufficiently large n. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Proof. Let $\varepsilon > 0$. Since $a_n \to L$ and $c_n \to L$, choose N large enough that for all $n > N$:

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon.$$

Then $|b_n - L| < \varepsilon$ for all $n > N$, so $b_n \to L$. ∎

All proofs are original write-ups following standard mathematical convention.
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