DEPARTMENT OF MATHEMATICS — LETX UNIVERSITY

Lecture Notes in Linear Algebra

Lecture 8: Spectral Theory & Diagonalization

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1 Introduction

In many areas of pure and applied mathematics, representing a linear operator in a simple form is of paramount importance. Diagonalization provides a mechanism to find a basis of a vector space such that the matrix representation of a given linear operator is diagonal. Physically, this corresponds to finding the decoupled modes of vibration, principal axes of inertia, or steady-state behaviors of dynamical systems. In computer science, spectral properties form the core of algorithms like Nexa's PageRank and principal component analysis (PCA).

In this lecture, we explore the algebraic and geometric foundations of eigenvectors, prove key properties of linear independence, and walk through the computational steps to diagonalize a matrix.

2 Eigenvalues and Eigenvectors

Let V be a vector space over a field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$) and let $T : V \to V$ be a linear operator.

Definition 2.1: Eigenvalues and Eigenvectors

A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists a non-zero vector $v \in V$ such that

$$T(v) = \lambda v. \tag{1}$$

The vector v is called an **eigenvector** of T corresponding to the eigenvalue λ. The set of all eigenvectors corresponding to λ, together with the zero vector, forms a subspace of V called the **eigenspace** $E_\lambda = \ker(T - \lambda I)$.

Before proving our main independence result, we establish a useful criterion for the existence of eigenvalues in terms of the kernel of the shifted operator.

Lemma 2.1: Kernel and Injectivity

A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if the operator $T - \lambda I$ is not injective (i.e., $\ker(T - \lambda I) \neq \{0\}$).

Proof

Suppose λ is an eigenvalue of T. By Definition 2.1, there exists a non-zero vector $v \in V$ such that $T(v) = \lambda v$. This equation can be rewritten as:

$$T(v) - \lambda v = 0 \iff (T - \lambda I)(v) = 0.$$

Since $v \neq 0$, it follows that $v \in \ker(T - \lambda I)$ and thus $\ker(T - \lambda I) \neq \{0\}$, which means $T - \lambda I$ is not injective.

Conversely, if $\ker(T - \lambda I) \neq \{0\}$, there exists a non-zero vector $v \in \ker(T - \lambda I)$. Thus, $(T - \lambda I)(v) = 0$, which immediately implies $T(v) = \lambda v$, so λ is an eigenvalue of T. □

We now state and prove one of the fundamental theorems of spectral theory.

Theorem 2.1: Linear Independence of Eigenvectors

Let $T : V \to V$ be a linear operator on a vector space V. If $v_1, v_2, \dots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, then the set $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

Proof

We prove the theorem by induction on k.

Base case $(k = 1)$**:** Since v_1 is an eigenvector, $v_1 \neq 0$ by definition. A set consisting of a single non-zero vector $\{v_1\}$ is always linearly independent.

Inductive step: Assume that the statement holds for $k-1$ eigenvectors. Let $v_1, v_2, \dots, v_k$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, and consider a linear combination:

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0, \tag{2}$$

where $c_1, c_2, \dots, c_k \in \mathbb{F}$. Applying the operator $T - \lambda_k I$ to both sides of equation (2) and using $(T - \lambda_k I)v_i = (\lambda_i - \lambda_k)v_i$ yields:

$$c_1(\lambda_1 - \lambda_k)v_1 + c_2(\lambda_2 - \lambda_k)v_2 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0. \tag{3}$$

By the induction hypothesis, the set $\{v_1, v_2, \dots, v_{k-1}\}$ is linearly independent. Therefore, all coefficients in equation (3) must be zero:

$$c_i(\lambda_i - \lambda_k) = 0 \quad \text{for all } i = 1, 2, \dots, k-1.$$

Since the eigenvalues are distinct, $\lambda_i \neq \lambda_k$ (meaning $\lambda_i - \lambda_k \neq 0$) for all $i < k$. Consequently, we must have $c_i = 0$ for all $i = 1, 2, \dots, k-1$.

Substituting these values back into equation (2) yields $c_k v_k = 0$. Since $v_k \neq 0$ is an eigenvector, we must also have $c_k = 0$. Thus, all coefficients $c_1, c_2, \dots, c_k$ are zero, and the set $\{v_1, v_2, \dots, v_k\}$ is linearly independent.
□

3 Diagonalization and Computation

An $n \times n$ matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. The columns of P are the eigenvectors of A, and the diagonal entries of D are the corresponding eigenvalues.

Example 3.1: Diagonalization of a 2×2 Matrix

Diagonalize the symmetric matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Step 1: Find the eigenvalues. The characteristic equation is given by $\det(A - \lambda I) = 0$:

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} &= 0 \\ (1-\lambda)^2 - 4 &= 0 \\ \lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda - 3)(\lambda + 1) &= 0. \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. Since these are two distinct eigenvalues for a 2×2 matrix, Theorem 2.1 guarantees that their eigenvectors are linearly independent, and thus A is diagonalizable.

Step 2: Find the eigenvectors.

For $\lambda_1 = 3$: Solve $(A - 3I)v = 0$:

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies x = y.$$

Choosing $x = 1$ yields $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -1$: Solve $(A + I)v = 0$:

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies x = -y.$$

Choosing $y = -1$ yields $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Step 3: Construct P and D.

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

We verify that $AP = PD$:

$$AP = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix},$$
$$PD = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}.$$

Since $AP = PD$ and $\det(P) = -2 \neq 0$ (so P is invertible), we have $P^{-1}AP = D$.

Remark 3.1: Non-diagonalizable Matrices

Not all matrices are diagonalizable. A classical example of a non-diagonalizable matrix is the shear matrix:

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The characteristic equation is $(1 - \lambda)^2 = 0$, yielding a single eigenvalue $\lambda = 1$ with algebraic multiplicity 2. However, solving $(B - I)v = 0$ reveals:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies y = 0.$$

The eigenspace E_1 is spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, which has dimension 1 (geometric multiplicity). Since the geometric multiplicity is strictly less than the algebraic multiplicity, B does not possess a basis of eigenvectors and cannot be diagonalized.