

# Problem Set 4 — Solutions

Applied Mathematics 204: Differential Equations

Meridian University — Department of Mathematics

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| <b>Instructor:</b>         | Dr. Nadia Farrow         |
| <b>Teaching Assistant:</b> | Owen Kessler             |
| <b>Assigned:</b>           | Week 9, Term 2           |
| <b>Due:</b>                | Friday, Week 10, 5:00 PM |
| <b>Total marks:</b>        | 40                       |

This solution set is provided for study purposes. Marking rubric notes appear in the margin beside each worked step.

## How to Read This Solution Set

Each problem below is restated in a shaded box, followed by a numbered worked solution. Small notes in the right margin, like the one beside this paragraph, reproduce the marking rubric used by the grading team: they show how many of the problem's points are attached to a given step and what a grader is checking for.

 — pt

Margin notes follow this format throughout: point value, then the grading criterion in one line.

| Symbol                                                                            | Meaning                                                                   |
|-----------------------------------------------------------------------------------|---------------------------------------------------------------------------|
|  | Marking-rubric note (margin) — points and criterion for the adjacent step |
|  | Final answer, boxed for quick reference                                   |
|  | Common mistake flagged at that step                                       |

**Partial credit policy.** Marks for a given step are awarded independently of later steps: an arithmetic slip in Step 2 does not forfeit the marks earned for correct setup in Step 1 or correct method in Step 3, provided the error is carried forward consistently. Full derivations are required — a correct final answer with no supporting work earns at most half of the marks allotted to the problem.

## 1 Problem 1 — Second-Order Linear ODE (10 points)

### Problem Statement

A damped oscillator is modelled by

$$y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

Find  $y(t)$  and classify the damping regime. [1]

**Step 1. Set up the characteristic equation.** Substituting  $y = e^{rt}$  gives

$$r^2 + 4r + 13 = 0.$$

 2 pt

Correct characteristic equation from the ODE coefficients.

**Step 2. Solve for the roots.** By the quadratic formula,

$$r = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i.$$

The discriminant is negative, so the roots are a complex-conjugate pair  $r = -2 \pm 3i$ , which places the system in the *underdamped* regime.

 3 pt

Correct roots *and* explicit statement of the damping regime (underdamped, since the discriminant  $< 0$ ).

**Step 3. Write the general solution.** For roots  $\alpha \pm i\beta$  with  $\alpha = -2$ ,  $\beta = 3$ , the general real solution is

$$y(t) = e^{-2t}(C_1 \cos 3t + C_2 \sin 3t).$$

**Step 4. Apply the initial conditions.** At  $t = 0$ :

$$y(0) = C_1 = 2.$$

Differentiating,

$$y'(t) = e^{-2t} [(-2C_1 + 3C_2) \cos 3t + (-2C_2 - 3C_1) \sin 3t],$$

so  $y'(0) = -2C_1 + 3C_2 = -1$ . With  $C_1 = 2$  this gives  $3C_2 = -1 + 4 = 3$ , hence  $C_2 = 1$ .

✓ **Final Answer**

$$y(t) = e^{-2t}(2 \cos 3t + \sin 3t), \quad \text{underdamped, } \omega_d = 3 \text{ rad/s.}$$

✎ **3 pt**

Correct product rule on  $y'(t)$  — the most common slip is dropping the  $-2$  factor from the envelope term when differentiating.

✎ **2 pt**

Final expression assembled correctly with both constants substituted; units/label for  $\omega_d$  not required but rewarded.

## 2 Problem 2 — Eigenvalues and Diagonalization (10 points)

**Problem Statement**

Let

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of  $A$ , and use them to write  $A = PDP^{-1}$ .

**Step 1. Characteristic polynomial.** Compute

$$\det(A - \lambda I) = \det \begin{pmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{pmatrix} = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10.$$

**Step 2. Solve for eigenvalues.** Factoring,

$$\lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2) = 0 \implies \lambda_1 = 5, \lambda_2 = 2.$$

**Step 3. Eigenvector for  $\lambda_1 = 5$ .** Solve  $(A - 5I)v = 0$ :

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies v_1 = 2v_2.$$

Taking  $v_2 = 1$  gives  $v^{(1)} = (2, 1)^T$ .

**Step 4. Eigenvector for  $\lambda_2 = 2$ .** Solve  $(A - 2I)v = 0$ :

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies v_1 = -v_2.$$

Taking  $v_2 = 1$  gives  $v^{(2)} = (-1, 1)^T$ .

✎ **2 pt**

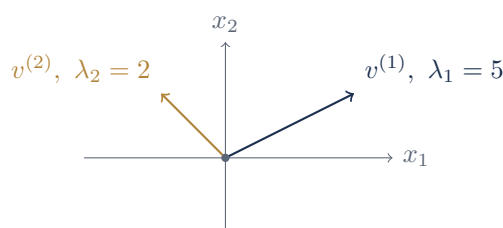
Determinant expanded correctly; sign error on the off-diagonal product is the most frequent deduction here.

✎ **3 pt**

Row reduction and free-variable choice; either eigenvector scaling is accepted as long as the direction is correct.

✎ **3 pt**

Same criterion as Step 3, applied to the second eigenspace.



## ✓ Final Answer

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}, \quad A = PDP^{-1}.$$

✎ 2 pt

$P$  built from eigenvectors as columns in the same order as the eigenvalues appear in  $D$  — a very common source of lost marks.

## 3 Problem 3 — Conditional Probability (10 points)

## Problem Statement

A component in a Nimbus sensor array is supplied by one of two vendors: Vendor A supplies 60% of units and 3% of those are defective; Vendor B supplies 40% of units and 5% of those are defective. A unit is drawn at random and found to be defective. What is the probability it came from Vendor B?

**Step 1. Define events and given probabilities.** Let  $B_A, B_B$  be the events “unit from Vendor A/B” and  $D$  the event “unit defective”.

$$\begin{aligned} P(B_A) &= 0.60, & P(B_B) &= 0.40, \\ P(D | B_A) &= 0.03, & P(D | B_B) &= 0.05. \end{aligned}$$

✎ 2 pt

Events correctly labelled and conditional probabilities matched to the right vendor.

**Step 2. Compute the total probability of a defect.** By the law of total probability,

$$\begin{aligned} P(D) &= P(D | B_A)P(B_A) + P(D | B_B)P(B_B) \\ &= (0.03)(0.60) + (0.05)(0.40) = 0.018 + 0.020 = 0.038. \end{aligned}$$

✎ 3 pt

Both terms present and correctly weighted; a common error is using  $P(B_A | D)$  in place of the prior  $P(B_A)$ .

**Step 3. Apply Bayes' theorem.**

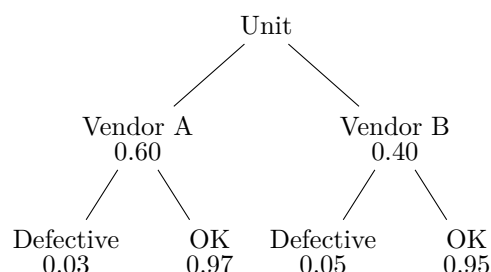
$$P(B_B | D) = \frac{P(D | B_B) P(B_B)}{P(D)} = \frac{(0.05)(0.40)}{0.038} = \frac{0.020}{0.038}.$$

✎ 3 pt

Correct Bayes' theorem set-up with the Step 2 result used as the denominator.

**Step 4. Simplify.**

$$P(B_B | D) = \frac{0.020}{0.038} \approx 0.526.$$



## ✓ Final Answer

$$P(\text{Vendor B} | \text{defective}) = \frac{0.020}{0.038} \approx 0.526 \text{ (52.6\%)}. \quad \text{Final probability reported to at least three significant figures with correct interpretation (posterior, not prior).}$$

✎ 2 pt

Final probability reported to at least three significant figures with correct interpretation (posterior, not prior).

## 4 Problem 4 — Constrained Optimization (10 points)

### Problem Statement

The Apex fabrication line uses a rectangular enclosure of fixed perimeter  $P = 40$  m for a cooling unit. Find the dimensions that maximize the enclosed area, and confirm the result is a maximum.

**Step 1. Set up the objective and constraint.** Let the sides be  $x$  and  $y$ . The area to maximize is  $A = xy$ , subject to

$$2x + 2y = 40 \implies y = 20 - x.$$

**Step 2. Reduce to a single-variable function.** Substitute  $y$  into  $A$ :

$$A(x) = x(20 - x) = 20x - x^2, \quad 0 < x < 20.$$

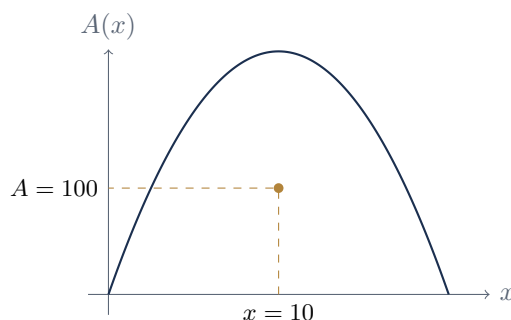
**Step 3. Find critical points.** Differentiate and set to zero:

$$A'(x) = 20 - 2x = 0 \implies x = 10.$$

**Step 4. Confirm it is a maximum.** The second derivative is

$$A''(x) = -2 < 0 \quad \text{for all } x,$$

so  $A(x)$  is concave down everywhere and  $x = 10$  is a global maximum on the interval. Then  $y = 20 - 10 = 10$ , so the optimal enclosure is a square.



### ✓ Final Answer

$$x = y = 10 \text{ m}, \quad A_{\max} = 100 \text{ m}^2 \quad (\text{the optimal rectangle is a square}).$$

2 pt

Constraint correctly solved for one variable and substituted in the next step (not left as a Lagrange multiplier problem, which is also accepted but not required here).

3 pt

Derivative taken correctly and set to zero; domain restriction  $0 < x < 20$  should be stated so the endpoints are ruled out.

3 pt

Second-derivative test (or a sign chart for  $A'$ ) explicitly invoked — a bare “this is the max” with no justification loses this mark.

2 pt

Both dimensions and the maximum area value reported, with the observation that the optimal shape is a square.

## Appendix: Formula Reference

The identities below were treated as known background for this problem set and did not need to be re-derived in any solution.

| Topic                        | Identity                                                                            |
|------------------------------|-------------------------------------------------------------------------------------|
| Complex characteristic roots | $y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$ for $r = \alpha \pm i\beta$ |
| 2×2 determinant              | $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$                       |
| Bayes' theorem               | $P(H \mid E) = \frac{P(E \mid H) P(H)}{P(E)}$                                       |
| Law of total probability     | $P(E) = \sum_i P(E \mid H_i) P(H_i)$                                                |
| Second-derivative test       | $f''(x_0) < 0 \Rightarrow x_0$ is a local maximum of $f$                            |

**Recommended further reading:** [1] for Problem 1, [3] for Problem 2, and [2] for Problem 3.

## References

- [1] William E. Boyce and Richard C. DiPrima. *Elementary Differential Equations and Boundary Value Problems*. Wiley, Hoboken, NJ, 11th edition, 2017.
- [2] Sheldon Ross. *A First Course in Probability*. Pearson, New York, NY, 10th edition, 2019.
- [3] Gilbert Strang. *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, MA, 5th edition, 2016.