Apex Institute of Technology

Department of Mathematics — MATH 204

Multivariable Calculus & Linear Algebra

Worksheet 04
Topic: Spectral Theorem & Vector Fields
Term: Fall 2026
Instructor: Dr. Elena Vance

Student Name: ____________________ **Student ID:** ______________ **Date:** __________

1. Linear Transformations & Eigenvalue Problems

Problem 1.1: Symmetric Matrix Diagonalization & Spectral Decomposition

Consider the symmetric matrix $A \in \mathbb{R}^{3\times 3}$ defined by:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

(a) Find the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$ and derive all eigenvalues.

(b) Construct an orthonormal basis of eigenvectors for $\mathbb{R}^3$ and matrix P such that $P^T AP = D$.

(c) Express A^5 using the spectral decomposition $A = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^T$.

Note: A real symmetric matrix $A = A^T$ has orthogonal eigenvectors for distinct eigenvalues. This guarantees diagonalizability over $\mathbb{R}$.

Detailed Derivation Step-by-Step:

Step 1: Compute eigenvalues. Subtract λ along the main diagonal:

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{bmatrix} = (2-\lambda)\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{bmatrix}$$

Expanding the remaining 2×2 determinant:

$$(4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5) \implies p_A(\lambda) = -(\lambda-2)^2(\lambda-5) = 0$$

Thus, the eigenvalues are $\lambda_1 = 5$ (mult. 1) and $\lambda_2 = \lambda_3 = 2$ (mult. 2).

Step 2: Eigenspace for $\lambda_1 = 5$.

$$(A - 5I)\mathbf{v} = 0 \implies -2x + y + z = 0,\ x - 2y + z = 0 \implies \mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step 3: Eigenspace for $\lambda_2 = 2$.

$$(A - 2I)\mathbf{v} = 0 \implies x + y + z = 0 \implies \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ \mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Algebraic Multiplicity
Multiplicity of λ as a root of $\det(A - \lambda I) = 0$. Geometric multiplicity satisfies $1 \leq \dim(\ker(A - \lambda I)) \leq$ alg. mult.

Step 4: Orthogonal Matrix P and Spectral Power Calculation.

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Using projection operators $P_1 = \mathbf{u}_1\mathbf{u}_1^T = \frac{1}{3}\mathbf{J}$ and $P_2 = I - \frac{1}{3}\mathbf{J}$ (where $\mathbf{J}$ is all ones):

$$A^5 = 5^5 P_1 + 2^5 P_2 = 3125\left(\frac{1}{3}\mathbf{J}\right) + 32\left(I - \frac{1}{3}\mathbf{J}\right) = 32I + 1031\mathbf{J} = \begin{bmatrix} 1063 & 1031 & 1031 \\ 1031 & 1063 & 1031 \\ 1031 & 1031 & 1063 \end{bmatrix}$$

Note: Gram-Schmidt process ensures orthogonality when eigenspaces have dimension ≥ 2. Here $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$ holds by construction.

2. Multivariable Calculus & Vector Field Integration

Problem 2.1: Green's Theorem & Circulation of Vector Fields

Let C be the boundary of the region R bounded by $y = x^2$ and $y = 4$, oriented counter-clockwise. Evaluate the line integral:

$$I = \oint_C \left((y^2 + e^{\sqrt{x}})\, dx + (3xy + \sin(y^2))\, dy\right)$$

Conservative Fields

A vector field $\mathbf{F} = \nabla f$ is conservative iff $\operatorname{curl} \mathbf{F} = 0$ on a simply connected domain. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.

Step-by-Step Application of Green's Theorem:

1. Compute component partial derivatives:

$$P(x, y) = y^2 + e^{\sqrt{x}} \implies \frac{\partial P}{\partial y} = 2y, \qquad Q(x, y) = 3xy + \sin(y^2) \implies \frac{\partial Q}{\partial x} = 3y$$

2. Formulate double integral over region R:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3y - 2y = y \implies I = \iint_R y\, dA = \int_{-2}^{2} \int_{x^2}^{4} y\, dy\, dx$$

3. Evaluate inner and outer integrals:

$$\int_{x^2}^{4} y\, dy = \left[\frac{y^2}{2}\right]_{x^2}^{4} = 8 - \frac{x^4}{2} \implies I = 2\int_0^2 \left(8 - \frac{x^4}{2}\right) dx = 2\left[8x - \frac{x^5}{10}\right]_0^2 = \frac{128}{5} = 25.6$$

Note: Green's Theorem transforms a closed line integral $\oint_{\partial R} P\, dx + Q\, dy$ into a double integral $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ over region R.

Key Result / Theorem Summary

Core Theorems of Multivariable Calculus in $\mathbb{R}^2$ and $\mathbb{R}^3$:

Theorem	Domain Relation	Integrands Formula
Fundamental Theorem	Curve C from A to B	$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$
Green's Theorem	Boundary $\partial R \to R \subset \mathbb{R}^2$	$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$
Divergence Theorem	Boundary $\partial R \to R \subset \mathbb{R}^2$	$\oint_{\partial R} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (\nabla \cdot \mathbf{F})\, dA$

3. Practice Problems & Conceptual Exercises

Problem 3.1: Coupled Linear System of Differential Equations

Consider the initial value problem for a coupled system of ODEs:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of A, and determine the matrix exponential e^{At}.

(b) Solve for the exact trajectory $\mathbf{x}(t)$ and analyze stability near the origin.

Submission Guidelines
Complete Problems 3.1 and 3.2 for peer grading. Show all intermediate steps and state relevant theorems explicitly.

Problem 3.2: Outward Flux Across a Closed Surface (Divergence Theorem)

Let $\mathbf{F}(x, y, z) = (x^3 + y)\mathbf{i} + (y^3 + z)\mathbf{j} + (z^3 + x)\mathbf{k}$. Compute the outward flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ across boundary S of sphere $B : x^2 + y^2 + z^2 \leq 1$.

Outline Solution for Problem 3.2: By Gauss's Divergence Theorem:

$$\begin{aligned} \Phi &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B (\nabla \cdot \mathbf{F})\, dV = 3 \iiint_B (x^2 + y^2 + z^2)\, dV \\ &= 3 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^1 \rho^4\, d\rho = 3(2\pi)(2)\left(\frac{1}{5}\right) = \frac{12\pi}{5} \end{aligned}$$

Note: Notice $\nabla \cdot \mathbf{F} = 3(x^2 + y^2 + z^2) = 3\rho^2$. Switch to spherical coordinates (ρ, ϕ, θ) to evaluate the triple integral efficiently.

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