ADVANCED LINEAR ALGEBRA & OPTIMIZATION

The Real Spectral Theorem

Orthogonal Diagonalization & Rayleigh Quotients

Prof. Alex Mercer

Department of Applied Mathematics · Nexa Institute of Technology

Spring Semester 2026

Lecture Agenda

Structure of Today's Discussion

01. Symmetric Operators
color cardborder!80

- Real symmetric matrices
- Real spectrum property
- Orthogonality of eigenspaces

02. The Spectral Theorem
color cardborder!80

- Formal theorem statement
- Spectral decomposition
- Projection operators

03. Step-by-Step Proof
color cardborder!80

- Existence of real roots
- Invariant complements
- Induction on dimension

04. Optimization & Geometry
color cardborder!80

- Rayleigh Quotient
- Quadratic form geometry
- Applications in Data Science

PART 01

Symmetric Matrices & Spectrum

Fundamental properties of real self-adjoint operators

Symmetric Matrices

Fundamental Definitions

Definition: Real Symmetric Matrix

color teal!30A matrix $A \in \mathbb{R}^{n \times n}$ is called **symmetric** if it equals its transpose:

$$A^T = A \quad \iff \quad a_{ij} = a_{ji} \quad \forall 1 \le i, j \le n.$$

Key Properties

color cardborder!80

- **Inner Product Symmetry:** For any $x, y \in \mathbb{R}^n$: $\langle Ax, y \rangle = (Ax)^T y = x^T Ay = \langle x, Ay \rangle$.
- **Self-Adjointness:** Under Euclidean inner product, symmetric matrices represent self-adjoint linear operators.
- **Spectrum:** We study the spectral properties of real eigenvalues and orthogonal eigenvectors.

Spectrum of Symmetric Matrices

Lemma 1: Reality of Eigenvalues

Lemma 1: All Eigenvalues are Real

color navy!30 Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Then all eigenvalues of A are real numbers, i.e., $\sigma(A) \subset \mathbb{R}$.

Proof Sketch

color cardborder!80 Let $\lambda \in \mathbb{C}$ be an eigenvalue with eigenvector $v \in \mathbb{C}^n \setminus \{0\}$, so $Av = \lambda v$.

Taking conjugate transpose:

$$v^* A^T = \bar{\lambda} v^* \quad \Longrightarrow \quad v^* A = \bar{\lambda} v^* \quad \text{(since } A^T = A\text{)}.$$

Multiply $Av = \lambda v$ on left by v^*: $v^* A v = \lambda v^* v = \lambda \|v\|^2$.
Multiply $v^* A = \bar{\lambda} v^*$ on right by v: $v^* A v = \bar{\lambda} v^* v = \bar{\lambda} \|v\|^2$.
Since $\|v\|^2 > 0$, we conclude $\lambda \|v\|^2 = \bar{\lambda} \|v\|^2 \implies \lambda = \bar{\lambda} \implies \lambda \in \mathbb{R}$. $\square$

Eigenspace Structure

Lemma 2: Orthogonality of Eigenspaces

Lemma 2: Orthogonal Eigenvectors

color navy!30 Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Eigenvectors corresponding to **distinct eigenvalues** are orthogonal.

Proof

color amber!40 Suppose $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$. Consider the inner product $\langle Av_1, v_2 \rangle$:

$$\begin{aligned} \langle Av_1, v_2 \rangle &= \langle \lambda_1 v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle, \\ \langle Av_1, v_2 \rangle &= \langle v_1, Av_2 \rangle = \langle v_1, \lambda_2 v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle. \end{aligned}$$

Subtracting yields $(\lambda_1 - \lambda_2)\langle v_1, v_2 \rangle = 0$. Since $\lambda_1 \neq \lambda_2$, we obtain $\langle v_1, v_2 \rangle = 0 \implies v_1 \perp v_2$. $\square$

PART 02

The Real Spectral Theorem

The foundational result of real matrix analysis

The Real Spectral Theorem

Main Statement

Theorem (Real Spectral Theorem)

color navy!30 Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Then:

1. A has n real eigenvalues (counting multiplicities): $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.
2. There exists an **orthonormal basis** $\{u_1, u_2, \ldots, u_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of A.
3. A is orthogonally diagonalizable:

$$A = Q\Lambda Q^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T,$$

where $Q = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ is orthogonal ($Q^T Q = I$) and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

PART 03

Highlighted Step-by-Step Proof

Constructive induction on spatial dimension

Proof of Spectral Theorem

Step 1: Base Case & Base Vector

Step 1: Existence of Initial Unit Eigenvector

color amber!40By Lemma 1, A has at least one real eigenvalue $\lambda_1 \in \mathbb{R}$.

- Let $v_1 \in \mathbb{R}^n \setminus \{0\}$ satisfy $Av_1 = \lambda_1 v_1$.
- Normalize to unit norm: $u_1 = \frac{v_1}{\|v_1\|_2}$, so $\|u_1\|_2 = 1$ and $Au_1 = \lambda_1 u_1$.

Inductive Setup

color cardborder!80

- We proceed by induction on dimension n.
- **Base Case ($n = 1$):** A 1×1 symmetric matrix $A = [a]$ is trivially diagonalized by $Q = [1]$ and $\Lambda = [a]$.
- **Induction Hypothesis:** Assume the result holds for all $(n-1) \times (n-1)$ symmetric matrices.

Proof of Spectral Theorem

Step 2: Invariance of Orthogonal Complement

Step 2: Invariance of $W = (\mathrm{span}\{u_1\})^\perp$

color amber!40 Define $W = \{x \in \mathbb{R}^n : \langle x, u_1 \rangle = 0\}$. Note that $\dim(W) = n - 1$.

Claim: W is invariant under A, i.e., $x \in W \implies Ax \in W$.

Verification of Invariance

color cardborder!80 Let $x \in W$. We compute the inner product of Ax with u_1:

$$\langle Ax, u_1 \rangle = \langle x, A^T u_1 \rangle = \langle x, Au_1 \rangle = \langle x, \lambda_1 u_1 \rangle = \lambda_1 \langle x, u_1 \rangle = 0.$$

Thus, $Ax \in W$. The restriction operator $A|_W : W \to W$ is symmetric on an $(n-1)$-dimensional space.

Proof of Spectral Theorem

Step 3: Induction Completion

Step 3: Application of Induction Hypothesis

color amber!40 By induction hypothesis, $A|_W$ has an orthonormal basis $\{u_2, u_3, \ldots, u_n\}$ of W consisting of eigenvectors of $A|_W$ (and hence of A).

Assembly of Full Orthonormal Basis

color cardborder!80 Since $u_1 \perp W$ and $\{u_2, \ldots, u_n\} \subset W$ is orthonormal:

$$\mathcal{B} = \{u_1, u_2, \ldots, u_n\} \quad \text{forms a full orthonormal basis of } \mathbb{R}^n.$$

Construct $Q = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$. Then $Q^T Q = I$, and:

$$Q^T A Q = Q^T \begin{bmatrix} \lambda_1 u_1 & \cdots & \lambda_n u_n \end{bmatrix} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \Lambda.$$

Thus $A = Q \Lambda Q^T$. $\square$

PART 04

Rayleigh Quotient & Geometry

Optimization properties and geometric interpretation

Rayleigh Quotient

Variational Characterization of Eigenvalues

Definition: Rayleigh Quotient

color teal!30 For a real symmetric matrix $A \in \mathbb{R}^{n\times n}$, the **Rayleigh Quotient** $R_A : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is:

$$R_A(x) = \frac{x^T A x}{x^T x}.$$

Min-Max Theorem (Courant-Fischer)

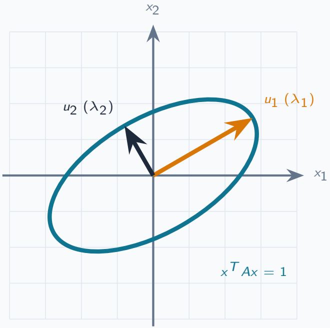
color navy!30 Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the ordered eigenvalues of A. Then:

$$\max_{x\neq 0} R_A(x) = \lambda_1 = R_A(u_1), \qquad \min_{x\neq 0} R_A(x) = \lambda_n = R_A(u_n).$$

Constrained optimization over orthogonal subspaces yields intermediate eigenvalues λ_k.

Geometric Intuition

Quadratic Forms & Ellipsoid Axes



Key Geometric Insights
color cardborder!80

- Level sets $x^T A x = c$ for $A \succ 0$ are **hyper-ellipsoids**.
- **Principal axes** align with orthogonal eigenvectors $u_1, \ldots, u_n$.
- Semi-axis lengths equal $r_i = \frac{\sqrt{c}}{\sqrt{\lambda_i}}$.

Matrix Factorization Summary

Comparative Overview

Factorization	Applicable Matrices	Form	Key Properties
Spectral	Real Symmetric ($A = A^T$)	$A = Q\Lambda Q^T$	Q orthogonal, Λ real diagonal
[3pt] **SVD**	Any Rectangular ($m \times n$)	$A = U\Sigma V^T$	U, V orthogonal, $\Sigma \geq 0$
[3pt] **Cholesky**	Symmetric Pos-Def ($A \succ 0$)	$A = LL^T$	L lower triangular ($l_{ii} > 0$)
[3pt] **QR**	Any Rectangular ($m \geq n$)	$A = QR$	Q orthonormal, R upper triangular

Practical Data Science Applications

color cardborder!80

- **Principal Component Analysis (PCA):** Diagonalize covariance matrix $C = \frac{1}{n}X^TX$.
- **Spectral Clustering:** Graph Laplacian eigenspace embedding $L = D - A$.

THANK YOU FOR LISTENING

Questions & Discussion

Next Lecture Preview

color cardborder!80Lecture 8: Singular Value Decomposition (SVD), Low-Rank Approximations,
and the Eckart-Young-Mirsky Theorem.

Contact: a.mercer@nexa.edu · Office Hours: Wed 14:00–16:00