Cohomological Methods in Arithmetic Geometry: Galois Representations, Period Rings, and p-Adic Hodge Theory

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To the memory of Professor Casimir Wray, who taught us to see the arithmetic in the geometry

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Abstract. This monograph develops the cohomological machinery underlying the modern theory of p-adic Galois representations attached to varieties over local and global fields. Building from Fontaine's period rings B_{dR}, B_{cris}, and B_{st}, we give a self-contained treatment of crystalline and semistable representations, the associated filtered (φ, N)-modules, and the comparison isomorphisms linking étale, de Rham, and crystalline cohomology of smooth proper varieties over p-adic fields. The middle chapters establish a syntomic-cohomological description of these comparison maps and prove a duality theorem for syntomic cohomology with compact supports (Theorem 3.2). The final chapters apply this machinery to bound Selmer groups attached to modular Galois representations, recovering and extending a class of results in the arithmetic of elliptic curves. Prerequisites are limited to a first course in algebraic number theory and the rudiments of scheme-theoretic algebraic geometry; all deeper input from p-adic Hodge theory is developed from first principles.

Contents

Preface

The arithmetic of Galois representations attached to algebraic varieties has, over the last four decades, grown from a collection of striking special cases into a coherent theory with its own internal logic. The unifying thread is p-adic Hodge theory: the study of how the p-adic étale cohomology of a variety over a p-adic field remembers, through a period ring, the de Rham and crystalline cohomology of the same variety. Fontaine's construction of the rings B_{dR}, B_{cris}, and B_{st} gave this remembering a precise algebraic form, and the subsequent work of Faltings, Tsuji, Niziol, Beilinson, Scholze, and many others turned the comparison isomorphisms conjectured by Fontaine into theorems of increasing generality and increasingly conceptual proof.

This book is an attempt to present a slice of that theory — crystalline and semistable representations, filtered (φ, N)-modules, and their syntomic-cohomological incarnation — at a level suitable for a second-year graduate student who has completed one course in algebraic number theory and one in scheme-theoretic algebraic geometry. We have not aimed for maximal generality; rather, we have tried to isolate the arguments that generalize cleanly from those that are special to small cases, and to give complete proofs of the former even where this has meant restricting the latter. Chapters 1–3 are foundational and can be read independently of the arithmetic applications in Chapters 4–6, which specialize the general theory to Selmer groups of modular forms.

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CHAPTER 1

Period Rings and Filtered Modules

1. The rings B_{dR} and B_{cris}

Let p be a prime and let K be a finite extension of $\mathbb{Q}_p$ with residue field k of cardinality $q = p^f$. Fix an algebraic closure $\overline{K}$ of K and write $G_K = \mathrm{Gal}(\overline{K}/K)$ for its absolute Galois group. Let $C = \widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$, equipped with its natural G_K-action.

Definition 1.1. The *tilt* of $\mathcal{O}_C$ is the ring

$$\mathcal{O}_C^\flat \;=\; \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p,$$

whose elements are sequences $(x_0, x_1, x_2, \dots)$ with $x_i \in \mathcal{O}_C/p$ and $x_{i+1}^p = x_i$ for all $i \geq 0$. The ring $\mathcal{O}_C^\flat$ is a perfect ring of characteristic p, complete for the valuation induced by that on $\mathcal{O}_C$.

Fontaine's ring B_{dR} is constructed from $\mathcal{O}_C^\flat$ via a Witt-vector and completion procedure that we recall in §1.3 below. For the moment we record only its essential structural properties.

Theorem 1.2 (Fontaine). *There is a complete discrete valuation ring B_{dR}^+ with residue field C, uniformizer t, and fraction field $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[t^{-1}]$, equipped with a continuous G_K-action and an exhaustive, separated, decreasing filtration $\{\mathrm{Fil}^i B_{\mathrm{dR}}\}_{i \in \mathbb{Z}}$ by $t^i B_{\mathrm{dR}}^+$, such that:*

(i) $(B_{\mathrm{dR}})^{G_K} = K$;
(ii) $\mathrm{gr}^0 B_{\mathrm{dR}} = C$, *and* $\mathrm{gr}^i B_{\mathrm{dR}} \cong C(i)$ *as a G_K-module for every $i \in \mathbb{Z}$, where $C(i)$ denotes C twisted by the i-th power of the cyclotomic character;*
(iii) *every de Rham representation of G_K becomes, after tensoring with B_{dR}, isomorphic to a trivial one of the same dimension.*

Inside B_{dR} sits the subring B_{cris}, obtained by adjoining divided powers of certain natural elements and imposing a convergence condition; it carries, in addition to the filtration and Galois action inherited from B_{dR}, a Frobenius endomorphism φ lifting the q-power Frobenius on $\mathcal{O}_C^\flat/p$.

Definition 1.3. A p-adic representation V of G_K is *crystalline* if

$$\dim_{K_0} \big(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\big)^{G_K} \;=\; \dim_{\mathbb{Q}_p} V,$$

where K_0 is the maximal unramified subextension of $K/\mathbb{Q}_p$. Equivalently, V is crystalline if the associated filtered φ-module $D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ has K_0-dimension equal to $\dim_{\mathbb{Q}_p} V$.

PROPOSITION 1.4. *The functor D_{cris} is exact and fully faithful on the category of crystalline representations of G_K, and its essential image is described exactly by the admissible filtered (φ, N)-modules of Definition 1.7 with $N = 0$.*

PROOF. Exactness follows because B_{cris} is $(\mathbb{Q}_p, G_K)$-regular in the sense of Fontaine's formalism: B_{cris} is a domain, $B_{\mathrm{cris}}^{G_K} = K_0$, and every nonzero element of B_{cris} fixed by G_K up to a $\mathbb{Q}_p$-scalar is already invertible in B_{cris}. These properties formally imply that the functor $V \mapsto D_{\mathrm{cris}}(V)$ takes short exact sequences of crystalline representations to short exact sequences of filtered φ-modules, and that a $\mathbb{Q}_p$-linear map of crystalline representations is determined by, and every admissible morphism of filtered φ-modules arises from, the induced map on B_{cris}-points. Full faithfulness is the content of Fontaine's comparison theorem, whose proof we defer to Chapter 3, where it is recovered as a corollary of the syntomic descent spectral sequence (Theorem 3.1). The identification of the essential image with admissible filtered φ-modules having $N = 0$ is the Colmez–Fontaine classification theorem, Theorem 1.8 below, specialized to the case of trivial monodromy. $\square$

2. Filtered (φ, N)-modules

To describe crystalline and, more generally, semistable representations linearly-algebraically, we introduce the following category.

DEFINITION 1.5. A *filtered (φ, N)-module over K* is a finite-dimensional K_0-vector space D equipped with:

(a) a σ-semilinear bijection $\varphi\colon D \to D$, where σ is the Frobenius of $K_0/\mathbb{Q}_p$;
(b) a $\mathbb{Q}_p$-linear nilpotent endomorphism $N\colon D \to D$ (the *monodromy operator*) satisfying $N\varphi = p\,\varphi N$;
(c) an exhaustive, separated, decreasing filtration $\{\mathrm{Fil}^i D_K\}_{i\in\mathbb{Z}}$ on $D_K := K \otimes_{K_0} D$ by K-subspaces.

For such a module one defines two integers measuring its size: the *Hodge–Tate weights* via the jumps of the filtration, and the *Newton slopes* via the elementary divisors of φ (after extending scalars to an algebraic closure of K_0).

DEFINITION 1.6. Let D be a filtered (φ, N)-module of K_0-dimension d.

$$t_N(D) := v_p\big(\det\varphi \mid D\big),$$
$$t_H(D) := \sum_{i\in\mathbb{Z}} i \cdot \dim_K \mathrm{gr}^i D_K.$$

DEFINITION 1.7. A filtered (φ, N)-module D is *weakly admissible* if $t_N(D) = t_H(D)$ and, for every (φ, N)-stable K_0-subspace $D' \subseteq D$ with the induced filtration on $K \otimes_{K_0} D'$, one has $t_N(D') \geq t_H(D')$.

THEOREM 1.8 (Colmez–Fontaine). *The functor D_{st} is an equivalence of categories between semistable representations of G_K and weakly admissible filtered (φ, N)-modules over K, with quasi-inverse V_{st}.*

This theorem, whose proof occupies §§2.3–2.6 below, is the linear-algebraic heart of the subject: it reduces questions about p-adic Galois representations — a priori governed by the enormous and poorly-understood group G_K — to questions about finite-dimensional linear algebra decorated with two commuting-up-to-scalar operators and a filtration.

EXAMPLE 1.9. The cyclotomic character $\chi\colon G_K \to \mathbb{Z}_p^\times$ is crystalline, with $D_{\mathrm{cris}}(\mathbb{Q}_p(1)) = K_0 \cdot e$ for a basis vector e satisfying $\varphi(e) = p^{-1}e$, and filtration jump $\mathrm{Fil}^{-1}K \cdot e = K \cdot e \supsetneq \mathrm{Fil}^0 K \cdot e = 0$. More generally $\mathbb{Q}_p(n)$ is crystalline for every $n \in \mathbb{Z}$, with $t_H = t_N = -n$.

REMARK 1.10. The reader familiar with classical Hodge theory will recognize t_H as an analogue of the weight of a Hodge structure and t_N as an analogue of the degree of the associated vector bundle in the sense of the Harder–Narasimhan formalism; weak admissibility is then exactly the semistability condition of that formalism, restricted to the module D itself rather than all its subobjects with the reversed inequality direction. This dictionary is made precise in §2.7.

CHAPTER 2

The Comparison Isomorphisms

1. Statement of the comparison theorems

Let X be a smooth proper variety over K. Comparison theorems relate the p-adic étale cohomology $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, viewed as a G_K-representation, to the de Rham cohomology $H^i_{\text{dR}}(X/K)$, viewed as a filtered K-vector space, and (when X has good reduction) to the crystalline cohomology of the special fiber, viewed as a φ-module over K_0.

Theorem 2.1 (C_{cris}, good reduction case). *Suppose X has good reduction, with smooth proper model $\mathcal{X}/\mathcal{O}_K$ and special fiber X_0 over k. For every $i \geq 0$ there is a canonical isomorphism of filtered φ-modules*

$$B_{\text{cris}} \otimes_{\mathbb{Q}_p} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \;\cong\; B_{\text{cris}} \otimes_{K_0} H^i_{\text{cris}}(X_0/W(k))[1/p],$$

compatible with the G_K-action on the left (through the first factor), the Frobenius $\varphi \otimes \varphi$, and the tensor-product filtration matching $H^i_{\text{dR}}(X/K)$ on the right after base change to K. In particular $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline representation with $D_{\text{cris}}\big(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\big) \cong H^i_{\text{cris}}(X_0/W(k))[1/p]$.

Theorem 2.2 (C_{st}, semistable reduction case). *Suppose X has semistable reduction. Then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is semistable, and $D_{\text{st}}\big(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\big)$ is canonically isomorphic, as a filtered (φ, N)-module, to the log-crystalline cohomology of the special fiber equipped with its monodromy operator N.*

Theorem 2.2 specializes to Theorem 2.1 exactly when $N = 0$, i.e. when the semistable reduction is in fact good; this is consistent with Example 1.9, where $N = 0$ on $D_{\text{cris}}(\mathbb{Q}_p(n))$ for every n.

2. Syntomic cohomology

The modern proofs of Theorems 2.1 and 2.2, due in various forms to Fontaine–Messing, Kato, Tsuji, and Niziol, proceed by constructing an intermediate cohomology theory — *syntomic cohomology* — that receives a period map from étale cohomology and admits an explicit description in terms of crystalline complexes. We develop this theory in full in Chapter 3; here we record only its shape.

Table 1. Cohomology theories compared in this monograph and the chapter developing each.

Cohomology theory	Linear-algebraic avatar	Chapter
Étale, $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$	p-adic G_K-representation	1
De Rham, $H^i_{\text{dR}}(X/K)$	Filtered K-vector space	1
Crystalline, $H^i_{\text{cris}}(X_0/W(k))$	φ-module over K_0	2
Log-crystalline	Filtered (φ, N)-module	2
Syntomic, $H^i_{\text{syn}}(X, r)$	Complex computing all of the above	3

CHAPTER 3

Syntomic Cohomology and Duality

1. The syntomic descent spectral sequence

THEOREM 3.1. *Let $X/\mathcal{O}_K$ be a smooth proper formal scheme with generic fiber X_K. For each $r \geq 0$ there is a spectral sequence*

$$E_1^{i,j} = H^j\big(X, \mathcal{S}(r)\big) \implies H^{i+j}_{\text{ét}}(X_{K,\overline{K}}, \mathbb{Q}_p(r)),$$

where $\mathcal{S}(r)$ denotes the syntomic complex of Definition 3.4, functorial in X and compatible with products.

Granting Theorem 3.1 and the finiteness results of §3.2, we obtain the main duality statement of this monograph.

THEOREM 3.2 (Syntomic duality). *Let $X/\mathcal{O}_K$ be smooth and proper of relative dimension n. There is a canonical, functorial trace map*

$$\mathrm{tr}\colon H^{2n+2}_{\mathrm{syn}}(X, n+1) \longrightarrow \mathbb{Q}_p,$$

and, for every $0 \leq i \leq 2n+2$, the cup-product pairing

$$H^i_{\mathrm{syn}}(X, r) \times H^{2n+2-i}_{\mathrm{syn},c}(X, n+1-r) \longrightarrow H^{2n+2}_{\mathrm{syn}}(X, n+1) \xrightarrow{\mathrm{tr}} \mathbb{Q}_p$$

is a perfect pairing of finite-dimensional $\mathbb{Q}_p$-vector spaces.

SKETCH OF PROOF. The proof proceeds in three stages, each occupying a section of Chapter 3. First (§3.5) one constructs the trace map using the syntomic avatar of the top Chern class together with the identification of H^{2n}_{cris} of the special fiber with $\mathbb{Q}_p(-n)$ furnished by resolution of singularities and the crystalline Poincaré duality theorem. Second (§3.6) one shows the pairing is compatible, via the comparison map of Theorem 3.1, with the classical Poincaré duality pairing on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$ constructed from the trace map in étale cohomology; this compatibility is checked on an open dense stratum of X and then propagated by proper base change. Finally (§3.7) perfectness of the pairing is reduced, via a filtration by the coniveau spectral sequence, to the case of X a point, where it is Tate local duality for G_K-modules. $\square$

COROLLARY 3.3. *For V crystalline with Hodge–Tate weights in $[0, n]$ and any finite set S of places of a number field F containing the places above p and the places of bad reduction, the Selmer group $\mathrm{Sel}_S(F, V)$ is finite-dimensional over $\mathbb{Q}_p$, with dimension bounded in terms of the local invariants $t_H(D_{\mathrm{cris}}(V_v))$ for $v \in S$.*

CHAPTER 4

Selmer Groups of Modular Galois Representations

1. Setup and the main bound

We now specialize the machinery of Chapters 1–3 to the two-dimensional Galois representation $\rho_f\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p)$ attached to a normalized cuspidal eigenform $f = \sum_{n\geq 1} a_n q^n$ of weight $k \geq 2$, level N coprime to p, and nebentypus ε.

PROPOSITION 4.1. *The restriction of ρ_f to $G_{\mathbb{Q}_p}$ is crystalline, with Hodge–Tate weights $\{0, k-1\}$, and $D_{\mathrm{cris}}(\rho_f|_{G_{\mathbb{Q}_p}})$ has characteristic polynomial of Frobenius equal to $X^2 - a_p X + \varepsilon(p)p^{k-1}$.*

PROOF. This is a restatement, in the language of Chapter 1, of the Eichler–Shimura relation together with the comparison theorem of Faltings for the Kuga–Sato variety attached to f (the case $k = 2$ being the classical case of an abelian variety with good reduction at p, to which Theorem 2.1 applies directly). The general weight case follows by realizing ρ_f inside $H^{k-1}_{\text{ét}}$ of the $(k-2)$-fold fiber product of the universal elliptic curve over the modular curve $X_1(N)$, and applying Theorem 2.1 to a smooth proper model of that fiber product, whose existence and good reduction away from Np is classical. $\square$

THEOREM 4.2. *Suppose $p \nmid N$ and ρ_f is irreducible. Then, for the Bloch–Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(\mathbb{Q}, \rho_f)$ attached to ρ_f,*

$$\dim_{\mathbb{Q}_p} \mathrm{Sel}_{\mathrm{BK}}(\mathbb{Q}, \rho_f) \;\leq\; \dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}_p, \rho_f) \;+\; \sum_{\ell \mid N} \dim_{\mathbb{Q}_p} H^0(\mathbb{Q}_\ell, \rho_f^\vee(1)),$$

where H^1_f denotes the crystalline (Bloch–Kato finite) local condition at p, computed via Proposition 4.1 as

$$\dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}_p, \rho_f) \;=\; \dim_{\mathbb{Q}_p}\big(D_{\mathrm{cris}}(\rho_f)/\mathrm{Fil}^0\big) \;=\; 1.$$

PROOF. Apply Corollary 3.3 with $V = \rho_f$, $n = k-1$, and S the set of places dividing $Np\infty$; the local dimension formula for $H^1_f(\mathbb{Q}_p, \rho_f)$ is the Bloch–Kato tangent space computation, specialized via Proposition 4.1 to the explicit filtration jump at 0 and $k-1$. The archimedean and tame-ramification terms are absorbed into the sum over $\ell \mid N$ by the local Euler characteristic formula, Proposition 4.6 below. $\square$

REMARK 4.3. When f corresponds to an elliptic curve $E/\mathbb{Q}$ (so $k = 2$ and $\rho_f = E[p^\infty]$ rationally), Theorem 4.2 recovers the classical bound on the p-Selmer group of E used in the standard approach to the Birch–Swinnerton-Dyer conjecture via Iwasawa theory, with $H^1_f(\mathbb{Q}_p, \rho_f)$ identified with the image of the Kummer map $E(\mathbb{Q}_p) \otimes \mathbb{Q}_p \hookrightarrow H^1(\mathbb{Q}_p, V_p E)$.

Notation

K	finite extension of $\mathbb{Q}_p$
K_0	maximal unramified subextension of $K/\mathbb{Q}_p$
G_K	absolute Galois group $\mathrm{Gal}(\overline{K}/K)$
C	completion of $\overline{K}$
$B_{\mathrm{dR}}, B_{\mathrm{cris}}, B_{\mathrm{st}}$	Fontaine's period rings
$D_{\mathrm{cris}}, D_{\mathrm{st}}$	Dieudonné-module functors
φ	(crystalline) Frobenius
N	monodromy operator
t_H, t_N	Hodge and Newton numbers of a filtered (φ, N)-module
$H^i_{\mathrm{syn}}(X, r)$	syntomic cohomology of weight r

Bibliography

[1] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, in *The Grothendieck Festschrift, Vol. I*, Progr. Math., vol. 86, Birkhäuser, Boston, MA, 1990, pp. 333–400.

[2] P. Colmez and J.-M. Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), no. 1, 1–43.

[3] G. Faltings, *Crystalline cohomology and p-adic Galois representations*, in *Algebraic Analysis, Geometry, and Number Theory* (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80.

[4] J.-M. Fontaine and W. Messing, *p-adic periods and p-adic étale cohomology*, in *Current Trends in Arithmetical Algebraic Geometry* (Arcata, CA, 1985), Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 179–207.

[5] J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque **223** (1994), 59–111.

[6] K. Kato, *Semi-stable reduction and p-adic étale cohomology*, Astérisque **223** (1994), 269–293.

[7] W. Niziol, *Semistable conjecture via K-theory*, Duke Math. J. **141** (2008), no. 1, 151–178.

[8] P. Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1, 77 pp.

[9] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), no. 2, 233–411.